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On approximation of high order singular perturbations

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Abstract

Let L be a non-negative self-adjoint operator in a Hilbert space $\mathcal{H}_0$ with inner product $\langle\cdot,\cdot\rangle_0$ and let φ be a singular element belonging to $\mathcal{H}_{-k-1}\backslash\mathcal{H}_{-k}$ with $k \geqslant 2$ (high order), where $\{\mathcal{H}_s\}_{s=-\infty}^{\infty}$ is the scale of Hilbert spaces associated with L in $\mathcal{H}$. For the formal singular perturbation of L generated by φ self-adjoint realizations $H(g)$ in a Pontryagin space are considered and approximations of these realizations by suitable smoother models are investigated. The realizations $H(g)$ are described by k real parameters $(g_s)_{s=1}^{k}$ and given in a Pontryagin space $\Pi(\bar{g})$ of the form $\mathcal{H}_0 \oplus \mathbb{C}^m \oplus \mathbb{C}^m$, which is equipped with an inner product having $m = \left[\frac{k}{2}\right]$ negative squares and depending on parameters $\bar{g} = (g_s)_{s=2}^{2m}$. The approximating model includes a sequence of variable Pontryagin spaces $\mathcal{K}(\gamma^{(n)})$ of the form $\mathcal{H}_0 \oplus \mathbb{C}^{k-1}$ and a sequence of self-adjoint operators $A(\gamma^{(n)})$ in $\mathcal{K}(\gamma^{(n)})$, parametrized by a sequence $(\gamma_s^{(n)})_{s=1}^{k}$ of k real parameters. Approximation of the realization $\Pi(\bar{g})$, $H(g)$ is deduced from the approximation of φ by smoother elements $\psi^{(n)}$ from $\mathcal{H}_{-2}$ and under the asymptotic behaviour of parameters $\gamma_s^{(n)} + \langle (L-\mu)^{-s}\psi^{(n)}, \psi^{(n)}\rangle_0 \to g_s$, ($\mu < 0$ is a normalization point). The approximation is described in terms of the generalized resolvent convergence of suitable realizations $A(\gamma^{(n)}) \to H(g)$. Examples and applications are discussed.

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1. Introduction

Let $\mathcal{H}_0$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle_0$ and let L be a non-negative self-adjoint operator in $\mathcal{H}_0$. Denote by $(\mathcal{H}_l)_{l=-\infty}^{\infty}$ the scale of Hilbert spaces associated with L and $\mathcal{H}_0$, see [4]: $\mathcal{H}_l$ is the Hilbert space dom $L^{l/2}$ equipped with the norm $\|f\|_l = \|(L+1)^{l/2} f\|_0$, and for $l < 0$, $\mathcal{H}_l$ is the completion of $\mathcal{H}_0$ with respect to the norm $\|\cdot\|_l$. In a natural way

$\mathcal{H}_l$ and $\mathcal{H}_{-l}$ are duals and the inner product can be generalized to a pairing $\langle f, g\rangle_0$ between the spaces $\mathcal{H}_l$ and $\mathcal{H}_{-l}$: $|\langle f, g\rangle_0| \leqslant \|f\|_l \|g\|_{-l}$, $f \in \mathcal{H}_n, g \in \mathcal{H}_{-l}$, and $\langle g, f\rangle_0 = \langle f, g\rangle_0^*$. For $\pm l, n = 1, 2, \ldots$, the operator $(L+1)^{-n/2}$ is an isometry from $\mathcal{H}_l$ to $\mathcal{H}_{l+n}$. Finally, $\mathcal{H}_l \hookrightarrow \mathcal{H}_n, l > n$, and the inclusion map is contractive and has a dense range. In this paper we consider the formal rank-1 singular perturbation

$$L_\alpha \equiv L_{\alpha,\varphi} = L + \alpha\langle\cdot\,,\varphi\rangle_0\varphi, \tag{1.1}$$

where the *generalized element* φ is an element of the space $\mathcal{H}_{-k-1}\backslash\mathcal{H}_{-k}, k = 0, 1, \ldots$. If $k = -1$, or, more generally, if $\varphi \in \mathcal{H}_0$ then perturbation (1.1) is called *regular*, otherwise it is called *singular*. For the values $k = 0, 1$ perturbation (1.1) is singular, but admits a one-parameter family of self-adjoint realizations in $\mathcal{H}_0$, see [1, 2] and references therein. In the case $k = 0$ perturbation (1.1) can be treated as the generalized sum of L and the perturbation and the parameter α plays the role of a coupling constant. However in the case $k = 1$ a realization L^g of (1.1) is constructed indirectly via the Berezin–Faddeev 'restriction-extension' method [6], where a true parameter g (renormalized coupling constant) distinguishing between different self-adjoint realizations should be fixed by using auxiliary reasoning. According to the renormalization point of view the parameter g is defined by a normalization condition with respect to a point $\mu \in \rho(L)$. In [6] an approximation for the $\delta(x)$-perturbation of the Laplacian in $\mathbb{R}^3$ by rank-1 perturbations was used as a reason for the choice of an appropriate parameter. For approximations of $\mathcal{H}_{-1}$- and $\mathcal{H}_{-2}$- perturbations we refer to [1, 2].

In this paper we focus on approximations of *high order singular* perturbations, that is, perturbations with $k > 1$. Then L_α is just a formal expression on $\mathcal{H}_0$ and the (one-parameter family of) self-adjoint realizations, that is, operators or relations, are possible in inner product spaces different from the original space. In the approach, initiated by Berezin [5], the inner product is indefinite. Following this way in [8, 26, 27] a family of self-adjoint realizations of L_α in a Pontryagin space with $m = \left[\frac{k}{2}\right]$ negative squares is described. For further developments and applications we refer to [7, 11, 12, 17, 25]. A different approach for the realization problem was proposed recently in [21, 22], where the possibility of a Hilbert space realization was observed. A relation between these approaches was analysed in [9].

Here we follow the first approach and deal with realizations in a Pontryagin space. According to [8, 11, 26] the realization depends in general on k real parameters $(g_s)_{s=1}^k$; the subset $\bar{g} = \{g_2, \ldots, g_{2m}\}$ describes the inner product in the Pontryagin space, which has the structure $\Pi(\bar{g}) = \mathcal{H}_0 \oplus \mathbb{C}^m \oplus \mathbb{C}^m$. The parameter g_{2m+1}, when k is odd, plays a role in the construction, and we denote $\hat{g} = \{g_2, \ldots, g_k\}$. The parameter $g \equiv g_1$ plays the role of a coupling constant and for fixed parameters of $\hat{g}$ marks the one-parameter family of self-adjoint operators $H^g(\hat{g})$ realizing the formal perturbation (1.1). Similar to $\mathcal{H}_{-2}$-perturbations, all these parameters are defined relative to a normalization point $\mu < 0$. The complete description of the realization is given in terms of the triple $\Pi(\bar{g})$, $S(\hat{g})$, $H^g(\hat{g})$, where $S(\hat{g})$ is a symmetric nondensely defined operator in $\Pi(\bar{g})$, which is a kind of lifting of the restriction $L|_{f\in\mathcal{H}_{k+1},\langle f,\varphi\rangle_0=0}$. The operators (the relation if $g = \infty$) $H^g(\hat{g}), g \in \mathbb{R}\cup\{\infty\}$ form the one-parameter family of self-adjoint extensions of $S(\hat{g})$.

Thus, the realization scheme for high singular $\mathcal{H}_{-k-1}$-perturbations follows again the 'restriction–extension' method, but in the Pontryagin space $\Pi(\bar{g})$. The function

$$Q_k(z) = \left\langle \frac{(z-\mu)^k}{(L-z)(L-\mu)^k}\varphi, \varphi\right\rangle_0 + \sum_{l=0}^{k-1} g_{j+1}(z-\mu)^l \tag{1.2}$$

is the key ingredient of these perturbations, which generalizes the Nevanlinna function $Q_1(z)$ corresponding to $k = 1$ and appearing in $\mathcal{H}_{-2}$-perturbations. But $Q_k(z)$ with $k > 1$ is the generalized Nevanlinna function from the class $\mathcal{N}_m$, see [18–20].

Approximation of high order singular perturbations was considered recently by Shvedov in [28] in a class of operators, which are closely related with rank-1 perturbations of L. In contrast to $\mathcal{H}_{-2}$-perturbation the approximation in [28] includes approximation of spaces. The approximating spaces and operators there have the following description. The set of parameters consists of k real numbers $\hat{\gamma} = \{\gamma_s\}_{k=1}^{k}, \gamma_k \neq 0$. Denote $\bar{\gamma} = \{\gamma_s\}_{k=2}^{k}$. Let $\mathcal{K}(\bar{\gamma})$ be the Pontryagin space $\mathcal{K}(\bar{\gamma}) = \mathcal{H}_0 \oplus \mathbb{C}^{k-1}$, equipped with the inner product $\langle \cdot\,, \cdot \rangle_{\mathcal{K}(\bar{\gamma})} = \langle \cdot\,, (I_0 \oplus G_\gamma) \cdot \rangle_{\mathcal{H}_0 \oplus \mathbb{C}^{k-1}}$, where $G_\gamma = (\gamma_{ij})_{i,j=1}^{k-1}$ is a $(k-1)\times(k-1)$ upper triangle Hankel matrix, whose entries γ_{ij} have the properties $\gamma_{ij} = \gamma_{i+j}$ if $i + j \leqslant k$ and $\gamma_{ij} = 0$ if $i + j > k$. For a fixed parameters from $\bar{\gamma}$ a one-parameter family of self-adjoint operators $A^{\gamma_1}(\bar{\gamma}), \gamma_1 \in \mathbb{R} \cup \{\infty\}$, in $\mathcal{K}(\bar{\gamma})$ is constructed. The operators $A(\hat{\gamma}) \equiv A^{\gamma_1}(\bar{\gamma})$ are determined by L and a vector $\psi \in \mathcal{H}_0$. Sequences of numbers $\gamma_s^{(n)}$, $s = \overline{1, k}$, and elements $\psi^{(n)}$ from $\mathcal{H}_0$ are taken in such way that $\psi^{(n)} \overset{n\to\infty}{\longrightarrow} \varphi$ in $\mathcal{H}_{-k-1}$ and

$$\gamma_s^{(n)} + \langle (L-\mu)^{-s} \psi^{(n)}, \psi^{(n)} \rangle_0 \overset{n\to\infty}{\longrightarrow} g_s. \tag{1.3}$$

The spaces $\mathcal{K}_n \equiv \mathcal{K}(\bar{\gamma}^{(n)})$ and operators $A_n \equiv A(\hat{\gamma}^{(n)})$ are taken for the approximating objects. In [28] it was proven that the sequence $\mathcal{K}_n$ strongly approximates the space Π, for $z \in \rho(H^g(\hat{g}))$ the sequence $(A_n - z)^{-1}$ strongly approximates the resolvent $(H^g(\bar{g}) - z)^{-1}$, and the corresponding evolution operators $U_n(t)$ strongly approximates $U(t) = \mathrm{e}^{\mathrm{i} H^g(\hat{g}) t}$ in the sense of approximation with variable spaces, see [16] and for the Pontryagin space approximation see [23, 24].

In this paper we investigate the approximation of high singular perturbation from a different point of view following mainly Krein's extension theory. This will allow us to apply results to the approximation of singular boundary problems. Our aim is to interpret the above mentioned Shvedov approximating model, but here we use generalized elements $\psi \in \mathcal{H}_{-2}$ rather than vectors of $\mathcal{H}_0$, in terms of the extension theory ingredients and compare it with the previous realization. Particularly, we associate with such model the function

$$Q_{1k}(z) = \left\langle \frac{(z-\mu)}{(L-z)(L-\mu)} \psi, \psi \right\rangle_0 + \sum_{l=0}^{k-1} \gamma_{l+1} (z-\mu)^k. \tag{1.4}$$

If $\gamma_k < 0$, $Q_{1k}(z)$ and $Q_k(z)$ are generalized Nevanlinna functions from the class $\mathcal{N}_m$, and they both have only at infinity a generalized pole of nonpositive type and multiplicity m [20]. We observe that under conditions (1.3) the sequence $\{Q_{1k}(z) | \hat{\gamma} = \hat{\gamma}^{(n)}\}$ converges to $Q_k(z)$ uniformly on compact subsets in $\mathbb{C}\backslash\mathbb{R}^+$. Therefore we treat the function $Q_{1k}(z)$ as an approximant of the function $Q_k(z)$. From this point the approximation of high singular perturbation looks like the approximation of an operator representation of the generalized Nevanlinna function $Q_k(z)$ by representations of the function $Q_{1k}(z)$.

Besides this introduction, there are three sections. In section 2 we describe approximations of generalized Nevanlinna functions, associated with high singular perturbations, and the corresponding models. In section 3 the approximations of high singular perturbations are considered. An example and an application to a boundary value problem for the Bessel equation are analysed in section 4.

2. Approximations of generalized Nevanlinna functions and models

2.1. Preliminary

We recall some notions and facts from the extension theory of symmetric operators in Pontryagin spaces; for more details we refer to [3, 13, 14, 18–20]. Let S be a symmetric (not necessarily densely defined) operator with defect indices (1, 1) in a Pontryagin space

$\mathcal{K}$, $\langle\cdot\,,\,\cdot\rangle$, and let A be a self-adjoint extension of S in $\mathcal{K}$; A can be an operator or a linear relation with a nonempty resolvent set $\rho(A)$. A linear relation A is a linear manifold in $\mathcal{K}^2 \equiv \mathcal{K} \oplus \mathcal{K}$. Through dom A and ran A we denote the domain and the range of A which are defined in a natural way. The set $A(0) = \{f \in \mathcal{K} \mid \{0, f\} \in A\}$ is called *the multivalued part of* A; A is the graph of an operator if and only if $A(0) = 0$. We identify an operator B with its graph $\{\{f, Bf\} \mid f \in \operatorname{dom} B\}$. The adjoint to a linear relation A is defined as $A^* = \{\{h, h'\} \in \mathcal{K}^2 \mid \langle h', f\rangle = \langle h, f'\rangle \text{for all} \{f, f'\} \in A\}$. The linear relation $A = A^*$ is called self-adjoint. The inverse linear relation is given by $A^{-1} = \{\{f, f'\} \in \mathcal{K}^2 \mid \{f', f\} \in A\}$. A point z belongs to the resolvent set $\rho(A)$ of A if $(A-z)^{-1}$ is the graph of a bounded operator in $\mathcal{K}$, see [3, 13].

A *defect function* $\varphi(z)$ for S and A is a holomorphic function on $\rho(A)$ with values in $\mathcal{K}$ and the properties $\varphi(z) \in \ker(S^* - z)$ and

$$\varphi(z) - \varphi(\zeta) = (z-\zeta)(A-z)^{-1}\varphi(\zeta), \qquad z, \zeta \in \rho(A). \tag{2.1}$$

Then the *Q-function* for S and A is a holomorphic function $N(z)$ on $\rho(A)$ which satisfies the relation

$$N(z) = N(z_0^*) + (z - z_0^*)\langle\varphi(z), \varphi(z_0)\rangle, \qquad z, z_0 \in \rho(A), \tag{2.2}$$

in particular, $N(z)^* = N(z^*), z \in \rho(A)$. This function is uniquely defined up to a real constant. If $z_0 \in \rho(A)$ is fixed (2.2) describes an *operator representation* of $N(z)$.

If $\mathcal{K}$ has κ negative squares and the *minimality condition* $\mathcal{K} = \overline{\operatorname{span}}\{\varphi(z)|z \in \rho(A)\}$ is satisfied, it belongs to the class $\mathcal{N}_\kappa$ of *generalized Nevanlinna functions with* κ *negative squares*. Recall that $\mathcal{N}_\kappa$ is the set of all functions $N(z)$ which are defined and meromorphic in $\mathbb{C}^+ \cup \mathbb{C}^-$ such that $N(z)^* = N(z^*)$ and the kernel $\frac{N(z)-N(\zeta)^*}{z-\zeta^*}$ with z, ζ belonging to the *domain of the holomorphy* $\rho(N)$ of $N(z)$, has κ negative squares. If $\kappa = 0$, the class $\mathcal{N}_0$ consists of all *Nevanlinna functions*.

Also, conversely, each function $N(z) \in \mathcal{N}_\kappa$ which is not a real constant is a Q-function for a symmetric operator S in some Pontryagin space $\mathcal{K}$ and a self-adjoint extension A of S, which are uniquely determined up to unitary equivalence if the minimality condition is satisfied. Namely for the self-adjoint relation A in its operator representation and the symmetric relation

$$S = \{\{f, f'\} \in A | \langle (f' - z_0^* f), \varphi(z_0)\rangle = 0\}.$$

The minimality also implies that $\mathcal{K}$ has κ negative squares and $\rho(A) = \rho(N)$. If $N(z) \in \mathcal{N}_\kappa$ is a Q-function for S and A in $\mathcal{K}$ and the minimality condition holds then S is a densely defined symmetric operator if and only if $N(z)$ has the properties

$$(a) \lim_{y\to+\infty} y \operatorname{Im} N(\mathrm{i}y) = +\infty, \qquad (b) \lim_{y\to\infty} y^{-1} N(\mathrm{i}y) = 0. \tag{2.3}$$

Let S be a closed symmetric relation with defect indices (1, 1) in a Pontryagin space $\mathcal{K}$, A be a fixed self-adjoint extension of S in $\mathcal{K}$ and let $\varphi(z)$ be a defect function and $Q(z)$ be a Q-function for S and A. Then the following Krein formula establishes a bijective correspondence between the resolvents of all self-adjoint extensions $\widetilde{A}$ of S in $\mathcal{K}$ (canonical extensions) and the real numbers $\tau \in \mathbb{R} \cup \{\infty\}$ [18, 19]:

$$(\widetilde{A} - z)^{-1} = (A-z)^{-1} - \frac{\langle\cdot\,, \varphi(z^*)\rangle}{Q(z)+\tau}\varphi(z), \qquad z \in \rho(A) \cap \rho((Q+\tau)^{-1}). \tag{2.4}$$

This formula enables a parametrization $\widetilde{A} = A^\tau$ of the extensions such that $A^\infty = A$.

In this paper we meet generalized Nevanlinna functions of two different types. The functions of the first type appear in the framework of high order singular perturbations; they are holomorphic in $\mathbb{C}\backslash\mathbb{R}$ and admit the irreducible representation

$$N(z) = (z-\mu)^{2m} N_0(z) + p_{2m-1}(z), \tag{2.5}$$

where $N_0(z)$ is a Nevanlinna function holomorphic on $\mathbb{C}\backslash\mathbb{R}^+$ and satisfying (2.3) and $p_{2m-1}(z)$ is a polynomial of degree $\leqslant 2m-1$ with real coefficients. This $N(z)$ belongs to the class $\mathcal{N}_m$. Particularly, $Q_k(z)$ (1.2) is such a function. The functions of the second type appear as a Q-functions of smooth approximating models; they are also holomorphic in $\mathbb{C}\backslash\mathbb{R}$ and admit the representation

$$N(z) = N_0(z) + p_{k-1}(z), \tag{2.6}$$

where $N_0(z)$ is a Nevanlinna function $\mathbb{C}\backslash\mathbb{R}^+$ and $p_{k-1}(z)$ is a polynomial of degree $k-1$ with real coefficients. This $N(z)$ belongs to the class N_κ. After the normalization $\mathrm{Re}N_0(z_0) = 0$ at a point $z_0 \in \rho(L)$ (2.6) becomes an irreducible representation, i.e. unique. The number κ is given in terms of the leading coefficient of $p_{k-1}(z)$, say p_{k-1}, by the formula [19, lemma 3.3]

$$\kappa = \begin{cases} \left[\frac{k-1}{2}\right] & \text{if } k \text{ even and } p_{k-1} > 0, \\ \left[\frac{k}{2}\right] & \text{otherwise.} \end{cases} \tag{2.7}$$

The function $Q_{1k}(z)$ (1.4) is an example of such a function.

2.2. *Approximation of generalized Nevanlinna functions*

For clarity we list briefly some of the results concerning perturbation (1.1) for the values of $k = -1, 0, 1$; see [2, 29].

(i) If $k = -1$, or, more generally, $\varphi \in \mathcal{H}_0$, L_α is a self-adjoint operator in $\mathcal{H}_0$ with $\mathrm{dom}\, L_\alpha = \mathrm{dom}\, L$, since it is a rank-1 perturbation.

(ii) If $k = 0$, the perturbation $\alpha\langle\cdot\,,\varphi\rangle\varphi$ is relatively form bounded with respect to the sesquilinear form of the operator L and the perturbed operator L_α can be determined either by using the form perturbation method, or as the generalized sum of L and the perturbation, see [2, 29].

In both cases (i) and (ii) the resolvent of L_α is given by the same formula which is of the form (2.4) with the substitutions $A = L \equiv L_0$, $\widetilde{A} = L_\alpha$, $\tau = \frac{1}{\alpha}$ and

$$Q(z) = \left\langle \frac{1}{L-z}\varphi, \varphi \right\rangle, \qquad \varphi(z) = (L-z)^{-1}\varphi.$$

(iii) In the case $k = 1$ perturbation (1.1) should be treated through the extension theory. The restriction $L_{\min} = L|_{\{u \in \mathcal{H}_{k+1} \cap \mathrm{dom} L | \langle u,\varphi\rangle = 0\}}$ with $k = 1$ is a symmetric operator in $\mathcal{H}_0$ with defect indices (1, 1). The perturbed operator is no longer uniquely defined. It is now interpreted as a self-adjoint extension of $L_{\min}$. These extensions can be parametrized by one real parameter $g \in \mathbb{R} \cup \{\infty\}$ either according to Krein's formula (2.4), identifying there $A = L$, $\widetilde{A} = L^g$, $z_0 = \mu < 0$, $\tau = g$, and the Q-function with

$$Q_1(z) := \left\langle \frac{z-\mu}{(L-z)(L-\mu)}\varphi, \varphi \right\rangle,$$

or as an appropriate restriction of $L^*_{\min}$. Here $\mu < 0$ is considered as a fixed point normalization; changing μ corresponds to changing the parameter g. In this way a one-parameter family of self-adjoint realizations L^g in $\mathcal{H}_0$ for the formal expression (1.1) is obtained. Note that both functions $Q(z)$ and $Q_1(z)$ are Nevanlinna functions holomorphic in $\mathbb{C}\backslash\mathbb{R}^+$; furthermore, in the singular cases (ii) and (iii) the functions $Q(z)$ and $Q_1(z)$ are specified by the asymptotic behaviour (2.3) along the imaginary axes, whereas in the case (i) of regular perturbations $\lim_{y\to+\infty} y\,\mathrm{Im}\, Q(\mathrm{i}y) < +\infty$ and $\lim_{y\to\infty} y^{-1}Q(\mathrm{i}y) = 0$. The asymptotic behaviour along the negative half-axes of these functions in the cases (i), (ii) and (iii) is specified as follows [15, 29]:

$$\begin{array}{llll} \text{(i)} & \lim_{x\to-\infty} Q(x) = 0, & \lim_{x\to-\infty} xQ(x) = -\|\varphi\|_0 & \text{if} \quad \varphi \in \mathcal{H}_0; \\ \text{(ii)} & \lim_{x\to-\infty} Q(x) = 0, & \lim_{x\to-\infty} xQ(x) = -\infty & \text{if} \quad \varphi \in \mathcal{H}_{-\infty}\backslash\mathcal{H}_l; \\ \text{(iii)} & \lim_{x\to-\infty} Q_1(x) = -\infty, & \lim_{x\to-\infty} \frac{1}{x}Q_1(x) = 0 & \text{if} \quad \varphi \in \mathcal{H}_{-2}\backslash\mathcal{H}_{-1}. \end{array} \tag{2.8}$$

In case (iii) the relation between the parameter g in L^g and the coupling parameter α in (1.1) cannot be established without additional assumptions. Let us consider how the function $Q(z) + \alpha^{-1}$ is transformed into $Q_1(z) + g$ from the approximation point of view, when $\varphi \in \mathcal{H}_{-2}\backslash\mathcal{H}_{-1}$ is approximated by a sequence of $\psi^{(n)} \in \mathcal{H}_0$. For $\psi \in \mathcal{H}_0$, $\mu < 0$ and $q(z, \psi) := \langle (L - z)^{-1}\psi, \psi\rangle_0$ we define

$$q_1^r(z, \psi) := q(z, \psi) - q(\mu, \psi) = \left\langle \frac{(z-\mu)}{(L-z)(L-\mu)}\psi, \psi \right\rangle_0 .$$

It is clear that $q_1^r(z, \psi)$ considered as a quadratic form in ψ is continuous with respect to $\|\cdot\|_{-2}$-norm. Evidently, $q(z, \psi) + \alpha^{-1} = q_1^r(z, \psi) + q(\mu, \psi) + \alpha^{-1}$. Consider a sequence $\psi^{(n)} \in \mathcal{H}_0$ approximating φ in $\mathcal{H}_{-2}$ and a sequence of real numbers $\alpha^{(n)}$ satisfying the condition $q(\mu, \psi^{(n)}) + (\alpha^{(n)})^{-1} \overset{n\to\infty}{\longrightarrow} g$. By continuity the sequence $q_1^r(z, \psi^{(n)})$ converges to $q_1^r(z, \varphi) = Q_1(z)$ for $z \in \rho(L)$, and, therefore the sequence $q(z, \psi^n) + (\alpha^{(n)})^{-1}$ approximates the function $Q_1(z) + g$. Also, the sequence $L_{\alpha^{(n)}}$ approximates the L^q in the strong resolvent sense [2, theorem 1.4.4].

If $k \geqslant 2$ the restriction $L_{\min}$ of L above is essentially self-adjoint in $\mathcal{H}_0$ and its closure coincides with L. We analyse what happens with the function $q(z, \psi)$, when $\psi = \psi^{(n)} \in \mathcal{H}_0$ and the sequence $\psi^{(n)}$ approximates $\varphi \in \mathcal{H}_{-k-1}\backslash\mathcal{H}_{-k}$, and also a generalization of such an approximation. For $\psi \in \mathcal{H}_0$, $\mu < 0$, and $q(z, \psi)$ as above we define for $l = 0, 1, 2, \ldots$ the functions $q_l^r(z, \psi)$ by the rule $q_0^r(z, \psi) \equiv q(z, \psi)$ and

$$q_l^r(z, \psi) := q(z, \psi) - \sum_{j=0}^{l-1} \frac{(z-\mu)^s}{s!}\frac{\mathrm{d}^s}{\mathrm{d}z^s} q(z, \psi)|_{z=\mu} = \left\langle \frac{(z-\mu)^l}{(L-z)(L-\mu)^l}\psi, \psi \right\rangle_0 ,$$

when $l = 1, 2, \ldots$, and call $q_l^r(z, \psi)$ the *l-regularization* of $q(z, \psi)$, as it is continuous in ψ relative to $\|\cdot\|_{-l-1}$-norm. Let $\varphi \in \mathcal{H}_{-k-1}\backslash\mathcal{H}_{-k}$, $m = \left[\frac{k}{2}\right]$, and

$$Q_0(z) := (z-\mu)^{-2m} q_k^r(z, \varphi) = \left\langle \frac{(z-\mu)^{k-2m}}{(L-z)(L-\mu)^k}\varphi, \varphi \right\rangle_0 .$$

Lemma 2.1. *The function $Q_0(z)$ is a Nevanlinna function having the asymptotic behaviour (2.3) and its asymptotic behaviour along the negative half-axis is given by*

$$(a_1)\ \lim_{x\to-\infty} xQ_0(x) = -\infty, \qquad (b_1)\ \lim_{x\to-\infty} Q_0(x) = \begin{cases} 0 & \textit{if } k = 2m, \\ -\infty & \textit{if } k = 2m+1. \end{cases} \tag{2.9}$$

The proof is obtained applying the results of (2.8) (ii) and (iii).

Corollary 2.2. *The function $Q_k(z)$ in (1.2) admits the irreducible representation (2.5) with the function*

$$N_0(z) = Q_0(z) (k = 2m), \qquad N_0(z) = Q_0(z) + g_k (k = 2m+1). \tag{2.10}$$

Particularly, $Q_k(z) \in \mathcal{N}_m$.

Assume $\psi \in \mathcal{H}_{-l-1}$, $l = 0, 1, \ldots, k-1$, and consider the functions

$$Q_{lk}(z) = q_{lk}(z, \psi) := q_l^r(z, \psi) + \sum_{s=0}^{k-1} \gamma_{s+1}(z-\mu)^s, \tag{2.11}$$

where the γ_s are real numbers. These $Q_{lk}(z)$ are generalized Nevanlinna functions holomorphic in $\rho(L)$. Also for a given set of real numbers $(g_s)_{s=1}^k$ we consider sequences of real numbers $\left(\gamma_s^{(n)}\right)_{s=1}^k$ and elements $\psi^{(n)} \in \mathcal{H}_{-l-1}$ satisfying the conditions

$$\begin{aligned} &\gamma_s^{(n)} \overset{n\to\infty}{\longrightarrow} g_s && \text{if} \quad s \leqslant l, \\ &\gamma_s^{(n)} + \langle (L-\mu)^{-s}\psi^{(n)}, \psi^{(n)}\rangle_0 \overset{n\to\infty}{\longrightarrow} g_s && \text{if} \quad l < s \leqslant k. \end{aligned} \tag{2.12}$$

Proposition 2.3. *Let a sequence $\psi^{(n)} \in \mathcal{H}_{-l-1}$, $l \leqslant k$, approximate $\varphi \in \mathcal{H}_{-k-1} \backslash \mathcal{H}_{-k}$ in $\mathcal{H}_{-k-1}$, and a sequence of real numbers $\gamma_s^{(n)}$, $s = \overline{1, k}$, satisfy the asymptotic conditions (2.12). Then for $z \in \rho(L)$ and the function $Q_k(z)$ given by (1.2)*

$$\lim_{n\to\infty} q_{lk}(z, \psi^{(n)})\big|_{\gamma_s = \gamma_s^{(n)}} = Q_k(z).$$

Proof. Let $z \in \rho(L)$ and $l \leqslant k$. Using the definition of the l- and k-regularizations of $q(z, \psi)$ and the equalities $\frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}z^s} q(z, \psi)|_{z=\mu} = \langle (L-\mu)^{-s-1}\psi, \psi\rangle_0$ we can write

$$q_l^r(z, \psi) = q_k^r(z, \psi) + \sum_{s=l}^{k-1} (z-\mu)^s \left\langle \frac{1}{(L-\mu)^{s+1}}\psi, \psi \right\rangle_0 .$$

Inserting this formula into (2.11) we obtain that the function $q_{lk}(z, \psi)$ is the sum of $q_k^r(z, \psi)$ and the polynomial

$$p(z) = \sum_{s=0}^{l-1} \gamma_{s+1}(z-\mu)^s + \sum_{s=l}^{k-1} \left(\gamma_{s+1} + \left\langle \frac{1}{(L-\mu)^{s+1}}\psi, \psi \right\rangle_0 \right) (z-\mu)^s.$$

$q_k^r(z, \psi)$ is continuous in ψ with respect to the norm $\|\cdot\|_{-k-1}$ and

$$\lim_{n\to\infty} q_k^r(z, \psi^{(n)}) = \left\langle \frac{(z-\mu)^k}{(L-z)(L-\mu)^k}\varphi, \varphi \right\rangle_0 .$$

By the asymptotic conditions (2.12) the coefficients p_s of polynomials after substitution $\psi = \psi^{(n)}$ and $\gamma_s = \gamma_s^{(n)}$ converge to the numbers g_{s+1}. Hence the convergence $q_{lk}(z, \psi^{(n)}) \overset{n\to\infty}{\longrightarrow} Q_k(z)$ follows. □

2.3. Realization of high order singular perturbation. m–model

Let $L \geqslant 0$ be a non-negative operator in a Hilbert space, $\varphi \in \mathcal{H}_{-k-1} \backslash \mathcal{H}_{-k}$, $k > 1$, and $m = \left[\frac{k}{2}\right]$. We describe shortly the operator realization of the formal singular perturbations (1.1) of L following mainly [8, 26, 27] and with slight changes due to the positivity of L [9, 12]. The realization depends in a generic way on k real parameters, which are specified relative to a normalization point from the resolvent set of L. We take $\mu \in \mathbb{R}^-$ for the normalization point and a set of parameters $\mathbf{g} = (g_s)_{s=1}^k$. Denote by $\hat{g}$ and $\bar{g}$ the subsets $\hat{g} = (g_s)_{s=1}^{k-1}$ and $\bar{g} = (g_s)_{s=2}^{2m}$ and set $g \equiv g_1$ for the parameter g_1 playing the role of a renormalized coupling. Sometimes we shall omit indicating irrelevant parameters. In the description below we associate with L and φ a self-adjoint relation H^∞ in a Pontryagin space Π and consider a suitable one-dimensional restriction S of H^∞. The realization of (1.1) is then represented by the family of all canonical self-adjoint extensions H^g of S.

Define the elements $\varphi_j := (L-\mu)^{-j}\varphi \in \mathcal{H}_{2j-k-1}\backslash\mathcal{H}_{2j-k}$, $j=\overline{1,k-1}$. The Pontryagin space $\Pi(\bar{g})$ with m negative squares is defined as the space $\Pi(\bar{g}) = \mathcal{H}_0 \oplus \mathbb{C}^m \oplus \mathbb{C}^m$, whose elements are vectors $F = (f, \mathbf{a}, \mathbf{b})$, $f \in \mathcal{H}_0, \mathbf{a}, \mathbf{b} \in \mathbb{C}^m$, and the inner product $\langle\cdot\,,\,\cdot\rangle = \langle\cdot\,, \mathbf{G}\cdot\rangle_{\mathcal{H}_0\oplus\mathbb{C}^m\oplus\mathbb{C}^m}$ is given by the Gram operator

$$\mathbf{G} = I_0 \oplus \begin{pmatrix} 0 & I_m \\ I_m & G \end{pmatrix}, \qquad G = (g_{ij})_{i,j=1}^m, \qquad g_{ij} := g_{i+j}. \tag{2.13}$$

Here I_0, I_m are the identity operators in $\mathcal{H}_0$ and $\mathbb{C}^m$ respectively. Hence the inner product is parametrized by the numbers $(g_s)_{s=2}^{2m}$ of the subset $\bar{g}$. Note that G in (2.13) is a Hankel matrix.

We assume that the parameters from $\hat{g}$ are fixed, set $\Pi \equiv \Pi(\bar{g})$ and $R_0(z) = (L-z)^{-1}$, and write $\mathbf{e}_j$ for the jth unit vector in $\mathbb{C}^m$, $j=\overline{1,m}$. Then H^∞ is the linear relation

$$H^\infty = \{(h + b_{m+1}\varphi_{m+1}, \mathbf{a}, \mathbf{b}), (Lh + \mu b_{m+1}\varphi_{m+1}, \mathbf{a}', \mathbf{b}')\},$$

$$\mathbf{a}' = (\mu a_1 + a_0)\,\mathbf{e}_1 + \sum_{j=2}^m (\mu a_j + a_{j-1} - g_{m+j}b_{m+1})\,\mathbf{e}_j, \qquad \mathbf{b}' = \sum_{j=1}^m (\mu b_j + b_{j+1})\mathbf{e}_j,$$

$$h \in \operatorname{dom} L, \qquad b_1 = 0, \qquad (b_i)_{i=2}^{m+1}, \qquad (a_j)_{j=0}^{m-1} \in \mathbb{C}^m, \qquad a_m = \langle h, \varphi_m\rangle + \widehat{g}_{2m+1}b_{m+1},$$

where $\widehat{g}_{2m+1} = g_{2m+1}$ if $k = 2m+1$ and $\widehat{g}_{2m+1} = \left\langle R_0^{2m+1}(\mu)\varphi, \varphi\right\rangle_0$ if $k=2m$. Hence H^∞ is a self-adjoint linear relation in Π and its multivalued part is the subspace $H^\infty(0) = \{c(0,\mathbf{e}_1,0), c \in \mathbb{C}\}$. It holds that $\rho(H^\infty) = \rho(L)$, $\sigma(H^\infty) = \sigma(L) \cup \{\infty\}$ and the compression of the resolvent $(H^\infty - z)^{-1}$ to $\mathcal{H}_0$ coincides with the resolvent $(L-z)^{-1}$. In this sense the relation H^∞ is a 'lifting' of L from $\mathcal{H}_0$ to Π. Note ([11]) that the neutral subspace $\mathcal{L} = 0 \oplus \mathbb{C}^m \oplus 0$ of Π is the root space of H^∞ at ∞; therefore, $z = \infty$ is a singular critical point of H^∞. From the description we see that $H^\infty = H^\infty(\hat{g})$ depends on the parameters $g_2, \ldots, g_k$ of $\hat{g}$.

We choose two vectors $u = (0,0,\mathbf{e}_1)$ and $w = (0, \mathbf{e}_1, 0)$ in Π. The symmetric operator S is chosen such that the defect subspace $\ker(S^* - \mu) = \operatorname{ran}(S-\mu)^\perp$ is spanned by the vector u. Hence S and its adjoint S^* in $\Pi_m(\bar{g})$ are given by

$$S = H^\infty \cap \{\{u, \mu u\}\}^*, \qquad S^* = H^\infty \dotplus \operatorname{span}\{\{u, \mu u\}\}.$$

For $z \in \rho(L)$ the vector function $\Phi(z) := (I + (z-\mu)R^\infty(z))u \in \ker(S^* - z)$ and satisfies (2.1); therefore, it is a defect function for S and H^∞. Explicitly,

$$\Phi(z) = \left((z-\mu)^m R_0(z)\varphi_m, \sum_{j=1}^m (z-\mu)^m d_j(z)\mathbf{e}_j, \sum_{j=1}^m (z-\mu)^{j-1}\,\mathbf{e}_j \right),$$

$$d_j(z) = (z-\mu)^{k-m-j}\langle R_0(z)\varphi_m, \varphi_{k-m}\rangle_0 + \sum_{l=1}^{k-m-j} (z-\mu)^{l-1} g_{m+j+l}.$$

By definition the Q-function $Q(z)$ associated with S and H^∞ in Π is determined by the defect function $\Phi(z)$ according to formula (2.2) with substitutions $\varphi(z) = \Phi(z)$ and $z_0 = \mu$. We require the normalization $Q(\mu) = g_1 (\equiv g)$ and obtain that $Q(z)$ is expressed by formula (1.2), i.e. $Q(z) = Q_k(z)$. By corollary 2.2 $Q(z) \in N_m$ and also admits representation (2.5) with the Nevanlinna function $N_0(z)$ (2.10).

The operators $H^g \equiv H^g(\hat{g})$, $g \in \mathbb{R}$ are defined as the canonical self-adjoint extensions of the symmetric operator S and they are restrictions of S^* in Π

$$H^g = S \dotplus \operatorname{span}\{u, \mu u - (g + Q_k(\mu))w\}. \tag{2.14}$$

The parameter $g \in \mathbb{R} \cup \{\infty\}$ distinguishes between the distinct canonical self-adjoint extensions of S, and every canonical self-adjoint extension of S coincides with one of the H^g. Among them only H^∞ is the multivalued extension. The family of extensions H^g, $g \in \mathbb{R} \cup \{\infty\}$, is by definition the realization of the formal expression (1.1) in Π. For complete parametrization we write $H^g(\hat{g})$ for them.

Krein's formula (2.4) applied to the resolvent of H^g takes the form

$$(H^g - z)^{-1} = (H^\infty - z)^{-1} - \frac{\langle \cdot\,, \Phi(z^*)\rangle}{Q_k(z)}\Phi(z), \qquad z \in \rho(L) \cap \rho(H^g). \tag{2.15}$$

2.4. Auxiliary $(1k)$*-model*

Here we describe a model associated with the function $Q_{1k}(z)$ (1.4), which can be used later as an approximating model (approximant) for the previous m-model of high order singular perturbations. We assume L is a non-negative operator in $\mathcal{H}_0$ as above, $\psi \in \mathcal{H}_{-2}\backslash\mathcal{H}_0$, and $\mu < 0$. Define the vector-function $\psi(z) = (L - z)^{-1}\psi$ and elements $\psi_s = (L - \mu)^{-s}\psi$, $s = \overline{1, k-1}$. As was mentioned, $Q_{1k}(z)$ is a particular case of functions (2.6) with concrete Nevanlinna function $N_0(z) = (z - \mu)\langle\psi(z), \psi_1(\mu)\rangle_0$ satisfying (2.3) and normalized by $N_0(\mu) = 0$. Therefore $Q_{1k}(z) \in \mathcal{N}_\kappa$ with κ determined by γ_k through (2.7). Our aim is to construct an operator model for the function $Q_{1k}(z)$ in the sense of representation (2.2), i.e. it is needed to find a symmetric operator S and its self-adjoint extension in a Pontryagin space $\mathcal{K}$ for which $Q_{1k}(z)$ is a Q-function, and then reconstruct the whole family of self-adjoint extensions of S. The $(1k)$-model below is a solution which is used later for approximations.

Consider a set of real numbers $\hat{\gamma} = \{\gamma_s\}_{k=1}^{k}$, $\gamma_k \neq 0$, separate $\gamma \equiv \gamma_1$ and denote $\bar{\gamma} = \{\gamma_s\}_{k=2}^{k}$. Let $\mathcal{K}$ be the space $\mathcal{K} = \mathcal{H}_0 \oplus \mathbb{C}^{k-1}$, equipped with the inner product $\langle\cdot\,,\cdot\rangle_{\mathcal{K}} = \langle\cdot\,, (I_0 \oplus G_\gamma)\cdot\rangle_{\mathcal{H}_0\oplus\mathbb{C}^{k-1}}$, where $G_\gamma = (\gamma_{ij})_{i,j=1}^{k-1}$ is the $(k-1) \times (k-1)$ upper triangle Hankel matrix with elements γ_{ij} satisfying $\gamma_{ij} = \gamma_{i+j}$ if $i + j \leqslant k$, and $\gamma_{ij} = 0$ if $i+j > k$. G_γ has a triangle structure with zeros under the antidiagonal and nonzero determinant $\det G_\gamma = (-1)^{[k-1/2]}\gamma_k^{k-1}$. The number κ of negative squares of the inner product coincides with the number of negative eigenvalues of G_γ and is determined by the formula (2.7) with the identification $p_{k-1} = \gamma_k$. Hence $\mathcal{K}$ is a Pontryagin space. Writing the elements of $\mathcal{K}$ as $F = (f, c_1, \ldots, c_{k-1})$ we introduce the vectors $\widehat{u} = (\psi_1(\mu), \mathbf{e}_1)$, $\widehat{u}_j = (0, \mathbf{e}_j)$, $j = \overline{1, k-1}$, and $\widehat{u}' = (\mu\psi_1(\mu), 0, 0)$. Consider in $\mathcal{K}$ the operator S given by

$$\operatorname{dom} S = \Bigg\{(f, \mathbf{c}) \in \mathcal{K} | f \in \operatorname{dom} L, c_1 = 0, c_j \in \mathbb{C}, j = \overline{2, k-1}, \\ \gamma_k c_k = -\langle h, \psi\rangle_0 - \sum_{j=2}^{k-1}\gamma_j c_j\Bigg\} \tag{2.16}$$

$$S(f, \mathbf{c}) = (Lf, \mathbf{c}'), \qquad c_s' = c_{s+1} + \mu c_s, \quad s = \overline{1, k-1}.$$

As L is closed and the functional $\langle\cdot\,, \psi\rangle_0$ is bounded on $\operatorname{dom} L$ S is a closed non-densely defined symmetric operator. The subspace $(\operatorname{ran}(S - \mu))^\perp$ is spanned by the vector $\widehat{u}$; hence S has defect indices $(1, 1)$. Among the self-adjoint extensions of S there is one A^∞ which is a linear relation with nontrivial multivalued part $A^\infty(0) = \{(0, c\,\mathbf{e}_{k-1}), c \in \mathbb{C}\}$:

$$\operatorname{dom} A^\infty = \big\{(f, \mathbf{c}) \in \mathcal{K} | f \in \operatorname{dom} L, c_1 = 0, (c_j)_{j=2}^{k} \in \mathbb{C}^{k-1}\big\}, \\ A^\infty = \{\{(f, \mathbf{c}), (Lf, \mathbf{c}')\}, \qquad c_s' = c_{s+1} + \mu c_s, s = \overline{1, k-1}, c_k' \in \mathbb{C}\}. \tag{2.17}$$

After an easy calculation we obtain its resolvent

$$(A^\infty - z)^{-1}(f, \mathbf{c}) = (R_0(z)f, \mathbf{c}'), \qquad c_s' = \sum_{i=1}^{s-1} c_{s-i}(z - \mu)^{i-1}, \quad s = \overline{1, k-1}. \tag{2.18}$$

It is seen from the last formula that $\rho(A^\infty) = \rho(L)$, $\sigma(A^\infty) = \sigma(L) \cup \{\infty\}$ and the compression of the resolvent $(A^\infty - z)^{-1}$ to $\mathcal{H}_0$ coincides with $(L - z)^{-1}$. Hence the linear relation A^∞ is a lifting of L from $\mathcal{H}_0$ to $\mathcal{K}$ similar to the relation H^∞ in subsection 2.4. Note ([10]) that the nondegenerate subspace $\mathcal{L} = 0 \oplus \mathbb{C}^{k-1}$ of $\mathcal{K}$ is the root space of A^∞ at ∞; therefore, $z = \infty$ is a regular critical point of A^∞. S is a one-dimensional restriction of A^∞: $S = \{\{f, f'\} \in A^\infty \mid \langle f' - \mu f, \widehat{u}\rangle_{\mathcal{K}} = 0\}$. All self-adjoint extensions of S, which are operators in $\mathcal{K}$, can be identified with the following linear relation:

$$A^\gamma = S \dotplus \operatorname{span}\{\widehat{u}, \widehat{u}' - \gamma \widehat{u}_{k-1}\}. \tag{2.19}$$

For the defect function $\Psi(z) := \widehat{u} + (z - \mu)(A^\infty - z)^{-1}\widehat{u}$ of S and A^∞ a short calculation with (2.17) leads to the formula

$$\Psi(z) = (\psi(z), 1, (z - \mu), \ldots, (z - \mu)^{k-2}),$$

which yields the equality $(z - \mu)\langle\Psi(z), \Psi(\mu)\rangle_{\mathcal{K}} + \gamma_1 = Q_{1k}(z)$. Therefore, the function $Q_{1k}(z)$ (1.4) is a Q-function for S and A^∞. With these functions the following formula specifies Krein's formula (2.4) for this model:

$$(A^\gamma - z)^{-1} = (A^\infty - z)^{-1} - \frac{\langle\cdot\,, \Psi(z^*)\rangle}{Q_{1k}(z)}\Psi(z), \qquad \gamma_1 = \gamma. \tag{2.20}$$

Indicating the dependence on parameters we will write $\mathcal{K}(\bar{\gamma})$, $S(\bar{\gamma})$ and $A^\gamma(\bar{\gamma})$. Thus the $(1k)$-model, which is given by the triple $\mathcal{K}(\bar{\gamma})$, $S(\bar{\gamma})$, $A^\gamma(\bar{\gamma})$, $\gamma \in \mathbb{R} \cup \{\infty\}$ with the described constituents, solves the problem we asked at the beginning of the subsection. We will take in section 3 the operators (linear relation if $\gamma = \infty$) $A^\gamma(\bar{\gamma})$ as approximant for the m-model.

Note that minimal models of the generalized Nevanlinna functions of the form (2.6) and more general functions with generalized pole at infinity were described in [10] in the framework of rank-1 perturbations in the Pontryagin space. In the model described above we followed in the opposite way and reconstructed a model in the Pontryagin space by data L, ψ associated with a smaller Hilbert space; also we do not require the minimality condition.

2.5. A relation between the models

For the given real numbers γ_s, $s = \overline{2, k}$ and the triple $\mathcal{H}_0$, L, ψ as above define the numbers g_s by

$$g_s := \gamma_s + \left\langle R_0^s(\mu)\psi, \psi\right\rangle_0, \qquad s = 2, 3, \ldots, k, \tag{2.21}$$

and consider the Ponrtyagin space $\Pi(\bar{g})$, introduced in subsection 2.3, parametrized now by these numbers g_s. The following mapping $V : \mathcal{K}(\bar{\gamma}) \to \Pi(\bar{g})$ was proposed in [28] (there it was denoted by Q):

$$\begin{aligned} &V(f, \mathbf{c}) = (h, \mathbf{a}, \mathbf{b}) : && h = f - \textstyle\sum_{j=1}^m c_j\psi_j, \\ &a_j = \langle h, \psi_j\rangle_0 + \sum_{l=m+1}^{k-j} \gamma_{l+j}c_l, && b_j = c_j, \qquad j = \overline{1, m}. \end{aligned} \tag{2.22}$$

Denote $\mathcal{R}(V) = \operatorname{ran} V$. Properties of V depend on $k = 2m$ or $k = 2m + 1$. In the case $k = 2m$ we consider the vector $v = \left(-R_0^m(\mu)\psi, -\sum_{j=1}^m g_{j+m}e_j, e_m\right)$ in $\Pi(\bar{g})$ and denote $\mathcal{V} = \{cv, c \in \mathbb{C}\}$. A simple calculation yields

$$\langle v, v\rangle_{\Pi(\bar{g})} = \left\langle R_0^{2m}(\mu)\psi, \psi\right\rangle_0 - g_{2m} = -\gamma_{2m} \neq 0. \tag{2.23}$$

Therefore if $k = 2m$, $\mathcal{V}$ is a nondegenerate subspace.

Lemma 2.4. *Let the sets of parameters $\bar\gamma$ and $\bar g$ be related as in (2.21). Then V maps $\mathcal{K}(\bar\gamma)$ isometrically in $\Pi(\bar g)$ and V is surjective if $k = 2m+1$, but* $(\operatorname{ran} V)^{\perp} = \mathcal{V}$ *is a nondegenerate one-dimensional subspace if $k = 2m$. In the last case*

$$\Pi(\bar g) = \mathcal{R}(V) \oplus \mathcal{V}. \tag{2.24}$$

Proof. That V is isometric under relation (2.21) can be checked by a calculation. When $k = 2m+1$ $\dim \mathbb{C}^{k-1} = \dim \mathbb{C}^m \oplus \mathbb{C}^m$ and the mapping in (2.22) is isomorphism. But in the case $k = 2m$ this is not true. We prove that $\mathcal{R}(V)^{\perp} = \mathcal{V}$. Assume the orthogonality condition $\langle V(h, \mathbf{c}), (f, \mathbf{a}, \mathbf{b})\rangle_{\Pi(\bar g)} = 0$ for arbitrary $(h, \mathbf{c}) \in \mathcal{K}(\bar\gamma)$. It is equivalent to the equations

$$f + b_m R_0^m(\mu)\psi = 0, \qquad b_j = 0,\ j = \overline{1, m-1}, \qquad a_j + g_{j+m} b_m = 0, \quad j = \overline{1, m}.$$

Hence $\mathcal{R}(V)^{\perp}$ coincides with the subspace $\mathcal{V}$. As $\mathcal{V}$ is nondegenerate $\Pi(\bar g)$ admits the orthogonal decomposition (2.24). □

Thus for $k = 2m+1$, V is a unitary mapping and for $k = 2m$, V is an isometry such that $\mathcal{V} = \ker V^*$ of the adjoint mapping $V^* : \Pi(\bar g) \to \mathcal{K}(\bar\gamma)$, which is surjective and is a partial isometry if $k = 2m$.

The mapping V and relations (2.21) allow us to get a correspondence between the first m-model $H^g(\bar g)$ in $\Pi(\bar g)$ determined by the triple $\mathcal{H}_0, L, \varphi$ and the second $(1k)$-model $A^{\gamma}(\bar\gamma)$ in $\mathcal{K}(\bar\gamma)$ determined by the triple $\mathcal{H}_0, L, \psi$. Observe that in the basic quantities of subsection 2.3 for $(H^{\infty} - z)^{-1}$, $\Phi(z)$ and $(H^g - z)^{-1}$ we are able to change $\varphi \in \mathcal{H}_{-k-1} \backslash \mathcal{H}_{-k}$ to $\psi \in \mathcal{H}_{-2} \backslash \mathcal{H}_0$. After making this substitution we denote the resulting expressions for these quantities correspondingly by $\widetilde{R}^{\infty}(z)$, $\widetilde{\Phi}(z)$ and $\widetilde{R}^g(z)$. Note that in the case $k = 2m$

$$\widetilde{R}^{\infty}(z)\mathcal{V} = 0, \qquad \widetilde{R}^g(z)\mathcal{V} = 0 \qquad \text{and} \qquad \langle \widetilde{\Phi}(z), v\rangle = 0. \tag{2.25}$$

Theorem 2.5. *Under relation (2.21) between the parameters from the sets $\bar\gamma$ and $\hat g$ and $z \in \rho(L) \cup \{z \mid Q_k(z) \neq 0\}$, the following relations hold:*

$$V(A^{\infty} - z)^{-1} = \widetilde{R}^{\infty}(z)V, \qquad V(A^g - z)^{-1} = \widetilde{R}^g(z)V, \qquad V\Psi(z) = \widetilde{\Phi}(z).$$

A similar result for the second relation was proved in [28, lemma 3.7], where the operator function $\widetilde{R}^g(z)$ was defined by a linear system. Our proof is based on a calculus with linear relations. We describe here only the basic idea of the proof, and omit calculations, as they are beyond the scope of this paper. Details will be published elsewhere.

The observation is that after the changing of φ to the smoother ψ in the m-model instead of the self-adjoint operators $H^g, g \in \mathbb{R}$ we obtain self-adjoint linear relations $\widetilde{H}^g \equiv \{H^g|\varphi \to \psi\}$. $\widetilde{H}^g$ is an operator if $k = 2m+1$, but in the case $k = 2m$ it is not an operator and has the one-dimensional multivalued part $\{0, \mathcal{V}\}$ independently of g. Similarly for $g = \infty$ we obtain the linear relation $\widetilde{H}^{\infty} \equiv \{H^{\infty}|\varphi \to \psi\}$ whose multivalued part is one dimensional if $k = 2m+1$, but in the case $k = 2m$ this is a two-dimensional subspace $\{0, \operatorname{span}\{v, w\}\}$ generated by the vectors v and $w = (0, \mathbf{e}_1, 0)$. The operator-function $\widetilde{R}^g(z)$, $g \in \mathbb{R} \cup \{\infty\}$ is just the resolvent of the linear relation of $\widetilde{H}^g$. This explains property (2.25). Next, by an algebraic calculation we check that $\{VF, VF'\} \in \widetilde{H}^g$ if $\{F, F'\} \in A^{\gamma}$ and relations (2.21) between the sets $\hat\gamma$ and $\hat g$ are satisfied. Then the third relation $V\Psi(z) = \widetilde{\Phi}(z)$ follows according to the first equality of theorem 2.5 and the simple equality $u = V\widehat{u}$ from the definitions of $\Phi(z)$ and $\Psi(z)$ in subsections 2.3 and 2.4.

3. Approximation of high order singular perturbations

We follow the definitions of [16] and [23, 24]. Let $\mathcal{H}$ and $\mathcal{H}_n$, $n = 1, 2, \ldots,$ be Hilbert spaces, P_n linear mappings from $\mathcal{H}$ onto $\mathcal{H}_n$. The spaces $\mathcal{H}_n$ approximate $\mathcal{H}$ if $\|P_n u\|_n \to \|u\|$, for all $u \in \mathcal{H}$. The sequence $\{u_n\}, u_n \in \mathcal{H}_n$, strongly approximates $u \in \mathcal{H} : u_n \xrightarrow{s} u$, if $\|u_n - P_n u\|_n \to 0$, when $n \to \infty$. If B_n, B are bounded linear operators in $\mathcal{H}_n, \mathcal{H}$, then B_n strongly approximates B: $B_n \xrightarrow{s} B$, if $B_n P_n u \xrightarrow{s} Bu$ holds for each $u \in \mathcal{H}$. The sequence $\{B_n\}$ weakly approximates B: $B_n \xrightarrow{w} B$, if $\langle B_n P_n u, P_n v\rangle_n \to \langle Bu, v\rangle$. If $\mathcal{D}$ is a dense linear manifold in $\mathcal{H}$ and P_n are mappings from $\mathcal{D}$ onto $\mathcal{H}_n$, then $\mathcal{H}_n$ approximates $\mathcal{H}$ if and only if P_n are uniformly bounded and $\langle P_n u, P_n v\rangle_n \to \langle u, v\rangle$.

Let $\mathcal{H}_n, \mathcal{H}$ be Hilbert spaces, P_n be mappings from $\mathcal{H}$ into $\mathcal{H}_n$ such that $\mathcal{H}_n$ approximates $\mathcal{H}$. If J_n, J are bounded and boundedly invertible self-adjoint operators in $\mathcal{H}_n$ and $\mathcal{H}$ such that $\mathcal{K}_n = (\mathcal{H}_n, J_n), \mathcal{K} = (\mathcal{H}, J)$ are Krein spaces and $J_n \xrightarrow{w} J$ then one says that $\mathcal{K}_n$ approximates $\mathcal{K}$. The approximation is called stable if, additionally, $\sup\left\| J_n^{-1} \right\|_n < \infty$. The following statement holds in the case of Pontryagin spaces [23, 24]: let $\mathcal{K}_n, \mathcal{K}$ be Pontryagin spaces, $\mathcal{D}$ be a dense linear manifold in $\mathcal{K}$ and P_n be linear mappings from $\mathcal{D}$ onto $\mathcal{K}_n$ such that P_n are uniformly bounded and

$$\text{(i) ind } K_n = \text{ind } K < \infty, \qquad \text{(ii) } \lim_{n\to\infty} \langle P_n u, P_n v\rangle_n = \langle u, v\rangle, \qquad u, v \in \mathcal{D}.$$

Then $\mathcal{K}_n$ stably approximates $\mathcal{K}$.

We assume L in $\mathcal{H}_0$, the realizations H^g in Pontryagin space $\Pi(\bar{g})$ of high order singular perturbation and a sequence $\psi^{(n)} \in \mathcal{H}_{-2}\backslash\mathcal{H}_0$, which approximates in $\mathcal{H}_{-k-1}$ the element $\varphi \in \mathcal{H}_{-k-1}\backslash\mathcal{H}_{-k}$, are given. The problem is to find sequences of Pontryagin spaces $\mathcal{K}_n$, mappings $P_n : \Pi(\bar{g}) \to \mathcal{K}_n$ and self-adjoint operators A_n in $\mathcal{K}_n$ such that the sequence $\mathcal{K}_n$ would approximate $\Pi(\bar{g})$ and the sequence of resolvents of A_n would strongly approximate the resolvent of $H^g(\hat{g})$. We will solve this problem taking for $\mathcal{K}_n$, A_n appropriate spaces and operators of the $(1k)$-model.

First, we substitute into ingredients of the $(1k)$-model the variable data $\psi^{(n)}$, $\left(\gamma_s^{(n)}\right)_{s=2}^k$ numbering $n = 1, 2, \ldots$ instead of the fixed data ψ, $(\gamma_s)_{s=2}^k$. Denote the variable objects induced by these substitutions as $\mathcal{K}_n = \mathcal{K}(\bar{\gamma}^{(n)})$, $A_n^\gamma = A^\gamma(\bar{\gamma}^{(n)})$, $\Psi_n(z)$. Then we consider the sequence of numbers $g_s^{(n)}$ given by

$$g_s^{(n)} = \gamma_s^{(n)} + \left\langle R_0^s(\mu)\psi^{(n)}, \psi^{(n)}\right\rangle_0, \qquad s = \overline{2, k}, \tag{3.1}$$

and assume that the finite limits $\lim_{n\to\infty} g_s^{(n)} = g_s$ exist, when $\psi^{(n)} \xrightarrow{n\to\infty} \varphi$. This is equivalent to condition (2.12) with $l = 1$. Introduce the intermediate Pontryagin spaces $\Pi_n \equiv \Pi(\bar{g}^{(n)})$ which were obtained from the space $\Pi(\bar{g})$ after changing $g_s \to g_s^{(n)}$, $s = \overline{2, k}$ and $\varphi \to \psi^{(n)}$, and define the inclusion mappings $\mathbf{j}_n : \Pi(\bar{g}) \to \Pi_n$, which map $\Pi(\bar{g}) \ni f \to f \in \Pi_n$ and then take the limit $n \to \infty$. It is clear that the mappings $\mathbf{j}_n$ are bounded and boundedly invertible and satisfy the estimate

$$\langle \mathbf{j}_n F, \mathbf{j}_n F'\rangle_{\Pi_n} = \langle F, F'\rangle_\Pi + O(d(\bar{g}^{(n)}, \bar{g})), \qquad \text{for all } F, F' \in \Pi, \tag{3.2}$$

where $d(\bar{g}, \bar{g}') = \max_{2\leqslant s\leqslant 2m} |g_s - g_s'|$. Also we define mappings $V_n : \mathcal{K}_n \to \Pi_n$, vectors v_n, and subspaces $\mathcal{V}_n$ as variable versions of the mapping V (2.22), the vector v, and the subspace $\mathcal{V}$, which were obtained after substitution of the above variable data. According to lemma 2.4 each V_n is a unitary mapping if $k = 2m + 1$, but $\mathcal{R}(V_n)^\perp = \mathcal{V}_n$ if $k = 2m$. Also decomposition (2.24) holds for each n: $\Pi_n = \mathcal{R}(V_n) \oplus \mathcal{V}_n$.

Similarly we define the variable operator functions $\widetilde{R}_n{}^\infty(z), \widetilde{R}_n{}^g(z)$, and vector functions $\widetilde{\Phi}_n(z)$ by substitution into the corresponding $\widetilde{R}^\infty(z)$, $\widetilde{R}^g(z)$, $\widetilde{\Phi}(z)$ variable quantities

$\psi = \psi^{(n)}$, $g_s = g_s^{(n)}$, $s = \overline{2, k}$ assuming relations (3.1) are satisfied. Then by theorem 2.5 the equalities

$$V_n\left(A_n^\infty - z\right)^{-1} = \widetilde{R}_n{}^\infty(z)V_n, \qquad V_n\left(A_n^g - z\right)^{-1} = \widetilde{R}_n{}^g(z)V_n, \qquad V_n\Psi_n = \widetilde{\Phi}_n \tag{3.3}$$

hold. Define the mappings $P_n : \Pi(\bar{g}) \to \mathcal{K}_n$ by the equalities $P_n = V_n^*\mathbf{j}_n$. If $k = 2m$, P_n has nontrivial $\ker P_n = \mathbf{j}_n^{-1}\mathcal{V}_n$. Note that this definition differs from the definition of the mappings P_n in [28].

The following limit equality holds:

$$\lim_{n\to\infty}\langle P_n F, P_n F'\rangle_{\mathcal{K}_n} = \langle F, F'\rangle, \text{ for all } F, F' \in \Pi(\bar{g}). \tag{3.4}$$

We prove the last equality. For $k = 2m + 1$ $P_n = V_n^*\mathbf{j}_n$ and V_n^* is a unitary mapping; convergence (3.4) follows immediately from estimate (3.2). Let $k = 2m$. We have

$$\langle P_n F, P_n F'\rangle_{\mathcal{K}_n} = \left\langle V_n^* P_{\mathcal{R}(V_n)}\mathbf{j}_n F, V_n^* P_{\mathcal{R}(V_n)}\mathbf{j}_n F'\right\rangle = \left\langle P_{\mathcal{R}(V_n)}\mathbf{j}_n F, \mathbf{j}_n F'\right\rangle.$$

According to (3.1) and the assumption about the sequences $g_s^{(n)}$ and $\psi^{(n)}\gamma_{2m}^{(n)} \to \infty$, when $n \to \infty$. Also for large $\gamma_{2m}^{(n)}$ the estimate

$$\left\langle P_{\mathcal{R}(V_n)}F_n, F_n'\right\rangle_{\Pi_n} = \langle F_n, F_n'\rangle_{\Pi_n} + O\left(1/\gamma_{2m}^{(n)}\right) \tag{3.5}$$

holds as a consequence of (2.23). These facts and the estimate (3.2) yield the limit

$$\lim_{n\to\infty}\left\langle P_{\mathcal{R}(V_n)}\mathbf{j}_n F, \mathbf{j}_n F'\right\rangle_{\Pi_n} = \langle F, F'\rangle.$$

This proves the claim.

Consider a canonical decomposition $\Pi(\bar{g}) = \Pi^+ \oplus \Pi^-$, where Π^- is an m-dimensional negative subspace, and define the norm in $\Pi(\bar{g})$ as the Hilbert norm on this decomposition. By assumption $\gamma_k^{(n)} + \left\langle R_0^k(\mu)\psi^{(n)}, \psi^{(n)}\right\rangle_0 \overset{n\to\infty}{\longrightarrow} g_k$; hence $\gamma_k^{(n)} < 0$ for large enough n. For these n the negative index ind $K_n = \left[\frac{k}{2}\right] = m$ according to (2.7) with $p_{k-1} = \gamma_k^{(n)}$. This and (3.4) imply that the sequence K_n stably approximates $\Pi(\bar{g})$ (see the last statement in subsection 3.1). We make this point more concrete and define a norm in the spaces $\mathcal{K}_n$. It follows from (3.4) that the subspace $P_n\Pi^-$ would be m-dimensional negative for large enough n and we take the space $K_n^- = P_n\Pi^-$ for the m-dimensional negative subspace of $\mathcal{K}_n$. For these n we define the norm in $\mathcal{K}_n$ as the Hilbert norm associated with the decomposition $\mathcal{K}_n = P_n\Pi^- \oplus (P_n\Pi^-)^\perp$.

The following theorem describes the approximation of the m-model by a sequence of $(1k)$-models which is close to the approximation results in [28, lemmas 3.8, 3.11, 3.15 and theorem 1].

Theorem 3.1. *Let the sequence $\psi^{(n)} \in \mathcal{H}_{-2}\backslash\mathcal{H}_0$ approximate $\varphi \in \mathcal{H}_{-k-1}\backslash\mathcal{H}_{-k}$ and conditions (2.12) with $l = 1$ hold. Then the sequence of $\mathcal{K}_n$ approximates $\Pi(\bar{g})$ and with $g \in \mathbb{R} \cup \{\infty\}$ and $z \in \rho(H^g(\hat{g})$*

$$\left\|\mathbf{j}_n^{-1}\widetilde{R}_n^g(z)\mathbf{j}_n F - (H^g(\hat{g}) - z)^{-1}F\right\|_{\Pi(\bar{g})} \overset{n\to\infty}{\longrightarrow} 0, \qquad F \in \Pi(\bar{g}), \tag{3.6}$$

and the sequence $\left(A_n^g - z\right)^{-1}$ strongly approximates $(H^g(\hat{g}) - z)^{-1}$:

$$\left\|\left(A_n^g - z\right)^{-1}P_n F - P_n(H^g(\hat{g}) - z)^{-1}F\right\|_{\mathcal{K}_n} \overset{n\to\infty}{\longrightarrow} 0, \qquad F \in \Pi(\bar{g}). \tag{3.7}$$

Proof. The first statement was explained before the theorem. For (3.6), which is similar to the results of [28, lemma 3.8, lemma 3.11], we use theorem 2.5 and Krein's formula (2.15) and also that the topology of $\Pi(\bar{g})$ is equivalent to the topology of the orthogonal sum $\mathcal{H}_0 \oplus \mathbb{C}^m \oplus \mathbb{C}^m$. By these results the coordinate transcription in the representation $\Pi(\bar{g}) = \mathcal{H}_0 \oplus \mathbb{C}^m \oplus \mathbb{C}^m$ of the operator-functions $\mathbf{j}_n^{-1}\widetilde{R}_n^g(z)\mathbf{j}_n$ and $(H^g(\hat{g}) - z)^{-1}$ are given by the same formulae, but with the only difference that for the first functions the data $\psi^{(n)}$, $g_s^{(n)}$, $s = \overline{1,k}$ stand instead of the fixed data φ, g_s, $s = \overline{1,k}$. As $R^g(z)$ depends on φ, g_s, $s = \overline{1,k}$ linearly and continuously in $\varphi \in \mathcal{H}_{-k-1}$ and, also, by assumption, the first variable data approximate the second ones, approximation (3.6) follows.

To prove (3.7) we set $H^g \equiv H^g(\hat{g})$ and write the square of the norm on the left as

$$\|X_n\|^2_{\mathcal{K}_n} = \langle X_n, X_n\rangle_{\mathcal{K}_n} + 2\|X_n\|^2_{\mathcal{K}_n^-}, \qquad X_n = \left(A_n^g - z\right)^{-1} P_n F - P_n(H^g - z)^{-1}F.$$

Then we use $P_n = V_n^* \mathbf{j}_n$ and the equality $\left(A_n^g - z\right)^{-1} V_n^* = V_n^* \widetilde{R}_n^g(z)$, which is adjoint to (3.3), and write

$$X_n = V_n^* \widetilde{R}_n^g(z)\mathbf{j}_n F - V_n^* \mathbf{j}_n (H^g - z)^{-1} F.$$

Using the fact that V_n^* is an isometry if $k = 2m+1$ and a partial isometry if $k = 2m$ we obtain for $k = 2m$

$$\langle X_n, X_n\rangle_{\mathcal{K}_n} = \left\langle P_{\mathcal{R}(V_n)} Y_n, Y_n\right\rangle_{\Pi_n}, \qquad Y_n = \widetilde{R}_n^g(z)\mathbf{j}_n - \mathbf{j}_n(H^g - z)^{-1}F;$$

the same formula, but with the projection $P_{\mathcal{R}(V_n)}$ replaced by the identity operator, holds in the case $k = 2m+1$. Then we use in the case $k = 2m$ estimate (3.5) and in both cases, k is even/odd, convergence (3.6) and obtain that $\langle X_n, X_n\rangle_{\mathcal{K}_n} \overset{n\to\infty}{\longrightarrow} \langle F, F\rangle_{\Pi(\hat{g})}$. To get the convergence of $\|X_n\|^2_{\mathcal{K}_n^-}$ we take an orthonormal basis $\{w_j\}_{j=1}^m$ in Π^- and the induced basis $\{P_n w_j\}_{j=1}^m$ in $\mathcal{K}_n^-$. Doing as before we get $\langle X_n, P_n w_j\rangle_{\mathcal{K}_n^-} \overset{n\to\infty}{\longrightarrow} \langle F, w_j\rangle_{\Pi^-}$ and $\langle P_n w_i P_n w_j\rangle_{\mathcal{K}_n^-} \overset{n\to\infty}{\longrightarrow} \delta_{ij}$ for the elements of the Gram matrix. Hence $\|X_n\|^2_{\mathcal{K}_n^-} \overset{n\to\infty}{\longrightarrow} \|F\|^2_{\Pi^-}$. □

4. Example: approximation related to the Bessel equation

We consider here an example of a high singular perturbation related to the Bessel differential expression $\ell_\nu y(x) = -y''(x) + \frac{\nu^2 - 1/4}{x^2} y(x)$, $x \in (0,\infty)$. In the case $0 < \nu < 1$ the corresponding minimal operator S in $L^2(\mathbb{R}^+)$ is symmetric and has defect indices (1, 1). We denote by L the self-adjoint extension of S which was uniquely determined so that its spectrum is absolutely continuous, $\sigma(L) = [0,\infty)$, and the functions $y(x,\lambda) = C(\lambda)x^{1/2}J_\nu(x\sqrt{\lambda})$, $\lambda \in [0,a)$ form the complete set of generalized eigenfunctions of L. Then the function

$$\varphi(x,z) = \sqrt{x}(-z)^{\frac{\nu}{2}} K_\nu(x\sqrt{-z}), \qquad K_\nu(\zeta) = \mathrm{i}\frac{\pi}{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu} H_\nu^{(1)}(\mathrm{i}\zeta), \tag{4.1}$$

where $H_\nu^{(1)}(\zeta)$ is the Hankel function of order ν, so $K_\nu(\zeta)$ is the modified Bessel function of the third kind (the MacDonald function), belongs to $\ker(S^* - z)$ and is the defect function for S and L. The function

$$Q(z) = -\frac{\pi}{2}\frac{(-z)^\nu}{\sin \pi\nu} \tag{4.2}$$

is a Q-function associated with S and L. Here the branch of $(-z)^\nu$ is chosen so that $(-z)^\nu = r^\nu \mathrm{e}^{\mathrm{i}\nu(\theta-\pi)}$ if $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, $0 < \theta < 2\pi$. If $0 < \nu < 1$ $Q(z)$ is a Nevanlinna function, which satisfies conditions (2.3). This is the classical result, see [30].

If $\nu \geqslant 1$ there is a unique self-adjoint realization L of ℓ_ν in $\mathcal{H}_0 = L^2(\mathbb{R}^+)$ which we call the Bessel operator. In the following we consider only the noninteger case, when $\nu > 1$ and $\nu \neq 2, 3, \ldots$. The function $Q(z)$ (4.2) has sense for these ν and belongs to the class N_m with $m = \left[\frac{\nu+1}{2}\right]$ negative squares. Also $Q(z)$ admits representation (2.5) with this m and also representation (1.2) with $k = [\nu] + 1$. This was shown in [12], where the realization of the Bessel expression in a Pontryagin space Π with $\operatorname{ind}\Pi = m$ was described. For

$z \in \rho(L)$ the function $\varphi(x,z) \in \mathcal{H}_{1-k}\backslash\mathcal{H}_{2-k}$ and the generalized element φ is identified with the generalized element $\varphi^B = (L-\mu)\varphi(\cdot\,,\mu)$, $\mu < 0$. The space Π is the Pontryagin space $\Pi(\bar{g})$ from subsection 2.3, where the parameters g_s, $s = \overline{2,[\nu]}$ should be given by $g_s = g_s^B := \frac{1}{(s-1)!}Q^{(s-1)}(\mu)$; $Q^{(l)}(z)$ stands for lth derivative of $Q(z)$. The corresponding realization, we call it the Bessel m-model, in $\Pi(\bar{g})$ is given by the family of self-adjoint operators $H^g(\hat{g})$ (2.14) with these data.

Next we consider a regular boundary problem associated with ℓ_ν and then a relation between regular and singular models. This will explain the appearance of the indefinite metric in singular problems.

Let $\nu > 1, \nu \neq 2, 3, \ldots,$ $\varepsilon > 0$ be a parameter and L be the Bessel operator in $\mathcal{H}_0$. Consider the symmetric restriction $L_{\min} = L\mid_{\{f\in \operatorname{dom} L | f(\varepsilon)=0\}}$. As it is known all self-adjoint extensions of $L_{\min}$ in $\mathcal{H}_0$ form a one-parameter family and are restrictions of the maximal operator $L_{\min}^*$ by the boundary condition

$$h'(\varepsilon+0) - h'(\varepsilon-0) = \alpha h(\varepsilon), \qquad h \in \operatorname{dom} L_{\min}^*, \qquad \alpha \in \mathbb{R}\cup\{\infty\}.$$

L corresponds to $\alpha = 0$. We denote other extensions by L_α. The L_α can be treated either as an $\mathcal{H}_{-1}$– perturbation of L, or as an $\mathcal{H}_{-2}$-perturbation of L_∞ [29]. In the first interpretation L_α is considered as singular perturbation (1.1) of L with $\psi_\varepsilon = \delta(x-\varepsilon)$. Making a rescaling we redefine this element as $\psi_\varepsilon = \beta\varepsilon^{-\frac{\nu+1}{2}}\delta(x-\varepsilon)$, where β is a real constant; it is equivalent to a rescaling of α in the boundary condition. A calculation with Bessel functions proves that the function

$$\psi_\varepsilon(x,z) = \beta\varepsilon^{-\nu}x^{1/2}(K_\nu(\varepsilon\sqrt{-z})I_\nu(x\sqrt{-z})\chi_{[0,\varepsilon]}(x) + I_\nu(\varepsilon\sqrt{-z})K_\nu(x\sqrt{-z})\chi_{[\varepsilon,\infty]}(x)),$$

where $\chi_{[a,b]}(x)$ denotes the characteristic function of interval $[a,b]$, is a defect function, see subsection 2.1 for the definition, and the function

$$Q_\varepsilon(z) = \beta^2\varepsilon^{-2\nu}I_\nu(\varepsilon\sqrt{-z})K_\nu(\varepsilon\sqrt{-z})$$

is a Q-function for $L_{\min}$ and L. $Q_\varepsilon(z) \in \mathcal{N}_0$ and satisfies (2.3) and (2.8)(ii). The function $\psi_\varepsilon(x,z)$ corresponds to $(L-z)^{-1}\psi_\varepsilon$ and $Q_\varepsilon(z) = \langle (L-z)^{-1}\psi_\varepsilon, \psi_\varepsilon\rangle_0$. Taking $\beta = 2^\nu\Gamma(\nu+1)$, assuming $z \in \mathbb{C}\backslash\mathbb{R}^+$ and using the series representations of the functions $I_\nu(\zeta)$, $K_\nu(\zeta)$ we obtain that

$$Q_\varepsilon(z) = p_{[\nu]}(z,\varepsilon) + Q(z) + \varepsilon^{2([\nu]-\nu+1)}z^{[\nu]+1}f_1(\varepsilon^2 z) + \varepsilon(-z)^{\nu+1}f_2(\varepsilon^2 z). \tag{4.3}$$

Here $Q(z)$ is function (4.2), $p_{[\nu]}(z,\varepsilon) = \sum_{s=0}^{[\nu]} p_s(\varepsilon)z^s$, with $p_s(\varepsilon) = p_s\varepsilon^{2s-2\nu}$, where the p_s are real numbers independent of ε, and $f_1(\zeta)$, $f_2(\zeta)$ are entire functions.

In (4.3), besides $Q(z)$, there are two generalized Nevanlinna functions

$$\widehat{Q}_\varepsilon(z) := Q_\varepsilon(z) - p_{[\nu]}(z,\varepsilon) \qquad \text{and} \qquad \widetilde{Q}_\varepsilon(z) := Q(z) + p_{[\nu]}(z,\varepsilon).$$

The function $\widetilde{Q}_\varepsilon(z) \in \mathcal{N}_m$ and is of the form (2.5), as $Q(z) \in \mathcal{N}_m$. It is relevant to an approximation problem considered in [17, 25]. But here we concentrate on the first function $\widehat{Q}_\varepsilon(z)$, which is just of the form (2.6). The asymptotic estimates for $\widehat{Q}_\varepsilon(z)$ and its $[\nu]$ derivatives

$$\widehat{Q}_\varepsilon(z) = Q(z) + O(\varepsilon^{2([\nu]-\nu+1)}), \qquad \widehat{Q}_\varepsilon^{(j)}(z) = Q^{(j)}(z) + O(\varepsilon^{2([\nu]-\nu+1)}), \qquad j = \overline{1,[\nu]}, \tag{4.4}$$

follow easily from formula (4.3). Take a sequence $\varepsilon_n \to 0$, when $n \to \infty$. From (4.4) it follows that the functions $\widehat{Q}_{\varepsilon_n}(z)$ with $z \in \mathbb{C}\backslash\mathbb{R}^+$ approximate the function $Q(z)$ (4.2) together with the first $[\nu]$ derivatives, when $n \to \infty$.

Next consider the function $\psi_\varepsilon(x,\mu)$. We observe that the function (4.1) $\varphi(x,\mu) \neq 0$ for $x > 0$, as $K_\nu(\zeta) \neq 0$ on $\mathbb{R}^+$. Using this we write $\psi_\varepsilon(x,\mu) = \varphi(x,\mu)\eta_\varepsilon(x)$ with

$$\eta_\varepsilon(x) = \beta\varepsilon^{-\nu}|\mu|^{-\nu/2}\left(\frac{K_\nu(\varepsilon\sqrt{|\mu|})I_\nu(x\sqrt{|\mu|})}{K_\nu(x\sqrt{|\mu|})}\chi_{[0,\varepsilon]}(x) + I_\nu(\varepsilon\sqrt{|\mu|})\chi_{[\varepsilon,\infty]}(x)\right).$$

The whole coefficient with the multiplier $\beta\varepsilon^{-\nu}|\mu|^{-\nu/2}$ in front of $\chi_{[\varepsilon,\infty]}(x)$ is estimated as $1 + O(\varepsilon^2)$, when $\varepsilon \to 0$, and the absolute value of the whole coefficient function (with the multiplier) in front of $\chi_{[0,\varepsilon]}(x)$ with $x \leqslant \varepsilon$ is estimated by the function $\beta|\mu|^{-\nu/2}|I_\nu(x\sqrt{|\mu|})|_{x\leqslant\varepsilon} = O(\varepsilon^\nu)$. As a result we obtain the estimate

$\|\psi_\varepsilon(\cdot,\mu) - \varphi(\cdot,\mu)\|_{1-k} \leqslant \|\varphi(\cdot,\mu)\chi_{[0,\varepsilon]}(\cdot)\|_{1-k} + o(\varepsilon)$, which implies the convergence $\psi_{\varepsilon_n}(x,\mu) \overset{n\to\infty}{\longrightarrow} \varphi(x,\mu)$ in $\mathcal{H}_{1-k}$. Hence we conclude that for $n \to \infty$ the sequence $\psi_{\varepsilon_n} = (L-\mu)\psi_{\varepsilon_n}(\cdot,\mu)$ approximates φ^B in $\mathcal{H}_{-k-1}$.

Next we apply the results of sections 2 and 3 and describe the corresponding operator contents of this approximation. Taking $k = [\nu] + 1$ and identifying

$$\begin{aligned}&\psi^{(n)} = \psi_{\varepsilon_n}, \qquad \psi^{(n)}(z) \equiv \psi_{\varepsilon_n}(\cdot,z), \qquad \psi_i^{(n)} = (L-\mu)^{-i+1}\psi_{\varepsilon_n}(\cdot,\mu)\\ &\gamma_{s+1}^{(n)} \equiv -\frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}z^s}p_{k-1}(z,\varepsilon_n)|_{z=\mu}, \qquad s = \overline{1,k-1},\end{aligned} \tag{4.5}$$

we get $\widehat{Q}_{\varepsilon_n}(z) - \widehat{Q}_{\varepsilon_n}(\mu) + g = Q_{1k}(z)|\psi = \psi^{(n)}, \hat{\gamma} = \hat{\gamma}^{(n)}, \gamma_1 = g$, where $Q_{1k}(z)$ is function (1.4), and the sequence of these functions approximates the defining function $\underline{Q(z) - Q(\mu) + g}$ $\underline{\text{of the}}$ Bessel m-model. Also the approximation of derivatives $\widehat{Q}^{(j)}_{\varepsilon_n}(z) \overset{n\to\infty}{\longrightarrow} \widehat{Q}^{(j)}(z)$, $j = \overline{1,k-1}$ implies the asymptotic conditions (2.12), where $l = 1$ and g_s are taken from the Bessel data $g_s = g_s^B$. Then we take for the variable spaces and operators the spaces $\mathcal{K}_n \equiv \mathcal{K}(\bar{\gamma}^{(n)})$ and the operators $A_n^g \equiv A^g(\bar{\gamma}^{(n)})$ of the $(1k)$-model of subsection 2.4 with data (4.5). Applying theorem 3.1 to this case we conclude that in the sense of the theorem the described spaces $\mathcal{K}_n$ and the operators A_n^g approximate $\Pi(\bar{g})$ and $H^g(\hat{g})$ of the Bessel m-model in the case $\nu > 1$, $\nu \neq 2, 3, \ldots$.

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